Theorem 46. *A line drawn on a hemisphere perpendicular to the line which bounds the hemisphere divides it into two congruent parts.*

Theorem 47. *Two perpendiculars erected to the boundary of a hemisphere at points subtending a quadrantal segment divide the hemisphere into four congruent tri-rectangular tri-quadrantal triangles and divide the entire surface into eight such triangles.*

Of these theorems the first follows as a corollary of Theorems 45 and 27; the second contains the term quadrantal segment which must be defined and justified. By Theorem 5 is established the fact that any segment AB mey be carried into the segment BA by a rotation about some point C such that AC is congruent to BO. The point C may be called the *middle* point of the segment AB. Moreover as BC must in a similar manner be congruent to CB, it follows that AC is congruent to CB. Hence the point C satisfies a definition for bisector of a segment similar to that given in Theorem 35 for the bisector of an angle. *The middle point C of a segment AB is a point such that AC is congruent to CB.* Application of this to the case of a semi-line OO' shows that there is a point C such that OC is congruent to CO'. Such segments may be called quadrantal segments or quadrants. It will be left to the reader to prove the evident theorem that all quadrantal segments are congruent.

Let the line l be the boundary of the hemisphere considered. Let A and B be two points such that AB is a quadrantal segment. Further, let the perpendiculars erected at A and B cut in O and cut the line l in A' and B', respectively. The segments BA', $A'B'$, and $B'A$ are quadrants as may be seen by applying to the semi-lines intercepted between A, A' and B, B' the theorem that all quadrants are congruent. Hence the triangles ABO, $BA'O$, $A'B'O$ are mutually congruent (Theorems 39 and 11). Hence the four corresponding angles at O are mutually congruent and therefore right angles. The triangles are tri-rectangular. Also the corresponding sides AO, BO, $A'O$, $B'O$ are congruent and consequently are quadrants. The hemisphere has been divided into four congruent tri-rectangular triangles of which each side is a quadrantal segment. That all such triangles are congruent and that there are eight formed on the surface of the sphere by three mutually perpendicular lines is left to the reader to prove.

Theorem 48. *If on a given line l a point A is moved into the antipodal point A' and the direction a, issuing from A and lying along l, is moved into the direction a', issuing from A and lying along l in the same order, the motion is involutory and each point of the line l is moved into its antipodal point.*

The proof of this theorem is similar to that of Theorem 34. Let B be any point of l and B' the antipodal point. Suppose that B were carried into B°, and B' into B'°. The motion is involutory: for repeating it brings A' and a' back to A and a and hence all points return to their original positions. If **M** denote the motion, we have

$$A, A', B, B', B^\circ, B'^\circ, [\mathbf{M}]\ A', A, B^\circ, B'^\circ, B, B'.$$

The order of the points ABB'° is the same as that of $A'B^\circ B'$. Hence both the

points B°, B'° lie in the same semi-line BB' and not in different semi-lines. But $B^\circ B'^\circ$ is derived from BB' by a motion and must be a semi-line. Hence the contradiction — unless B' and B° coincide. The theorem is proved.

Theorem 49. *All the perpendicular erected to a given line meet in a pair of antipodal points.*

Let l be the given line, AA' and BB' two perpendiculars to it intersecting in the points O, O'. Let CC' be any other perpendicular. Consider the hemisphere containing O. Move C into C' and the direction CA in the direction $C'A'$. By Theorem 46 the portion of the surface between the perpendicular CC' and one half of the line l will be carried into the region bounded by CC' and the other half of the line l. Hence if O does not lie on CC' it will have passed from one side of it to the other. But by Theorem 48 the points A and A' will have interchanged their positions. The right angle at A will have taken the place of the congruent right angle at A' and vice versa. The same interchange will have taken effect in the case of B and B'. Hence the perpendiculars AA' and BB' will have remained unchanged as a whole and the point O cannot have passed from one side of CC' to the other. Hence the point O lies on the line CC', which was any perpendicular to the line l, and the theorem is proved.

Definition: *The points in which the perpendiculars to a given line meet are called the poles of that line.*

Theorem 50. *Every line drawn through the pole of a given line cuts the given line at right angles.*

Theorem 51. *If a quadrantal segment OA rotates about the point O the extremity A describes a line to which the rotating segment is always perpendicular.*

Definition: *The line described by the moving end of a quadrant which rotates about one end O is called the polar of the point O.*

Theorem 52. *If two semi-lines OAO' and OBO' meet in the two points O and O', the angles formed at O and O' are congruent if the corresponding sides lie on different semi-lines, symmetrical if they lie on the same semi-lines.*

Theorem 53. *From any point without a given line, other than the pole of the line, one and only one line can be drawn perpendicular to the given line.*

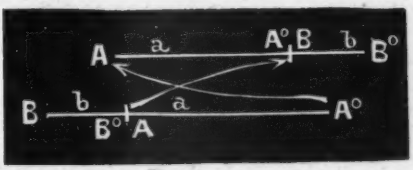
Theorem 54. *Two right triangles which have the hypothenuse and an acute angle of the one congruent to the hypothenuse and an acute angle of the other are congruent.*

Of these five theorems the first is an immediate corollary of Theorem 49, and the second may be proved from these two by showing that the line passed through any two positions of the point A is the required locus and satisfies the stated conditions. Theorem 52 is readily demonstrated by drawing the polar line of O, thus obtaining two bi-rectangular triangles with a common base (apply Theorems 5 and 10 or 11). To find one perpendicular to a given line and passing through a given point draw the line connecting the point to the pole of the line. There cannot be more than one, because two perpendiculars to a line intersect only in the poles of the line. The last theorem is proved by applying the congruent hypothenuses and allowing the congruent angles to coincide. Theorem 53 will then establish the uniqueness of the other side.

To take up the proof that to any triangle there exists a symmetrical triangle we shall prove the following lemma, which will be so important when we come to discuss the idea of measure that we may denote it as a theorem and refer back to it at that time.

Definition of the sum of two segments: *Given the two segments AA°, represented by a, and BB°, represented as b. The sum $a+b$ is the segment AB° obtained by laying BB° down along the line AA° produced so that B, the initial point of the second segment, coincides with A°, the final point of the first segment.*

Theorem 55. *The sum, $a+b$, of two segments a and b is congruent to the sum, $b+a$, of the segments taken in inverse order.*



Let a be designated as AA°, and b as BB°. Then $a+b$ is $A(A^\circ B)B^\circ$ where A° and B are coincident points. In like manner $b+a$ is $B(B^\circ A)A^\circ$ where B° and A coincide. Apply A° of the second to A of the first in such a way that the direction from A° to A falls along the direction from A to A° in the first figure. From Theorem 5 it follows that A falls on A°. Then $B^\circ B$ of the second figure will take the direction BB° of the first and B will fall on B°. The segments AB° and $A^\circ B$ coincide and are therefore congruent. Hence AB° and BA° are congruent and the theorem is proved.

Theorem 56. *If two segments c and d be respectively added to a given segment x and yield congruent segments $c+x$ and $d+x$, the segments c and d are congruent.*

In case they are added on the same end of the segment x the theorem is evident at once. In case they are added on opposite ends a proof similar to that given above and dependent on Theorem 5 may be given.

Theorem 57. *To a given triangle there may be found a symmetrical triangle.*

Let ABC be the given triangle. Produce the sides AB and AC until they meet in the point A' antipodal to A. In like manner produce the other pairs of sides and obtain the points B' and C'. The triangles ABC and $A'B'C'$ are symmetrical. For in the first place AA' and BB' are semi-lines with the common segment $B'A$. Hence by Theorem 56 the segments AB and $A'B'$ respectively which added to the same segment $B'A$ yield the congruent semi-lines, are themselves congruent. In like manner BC and CA are congruent to $B'C'$ and $C'A'$. And in the second place the angle BAC at A is either congruent or symmetrical to the angle $B'A'C'$ at A' (Theorems 34 and 52). Now AB and $A'B'$ are directed in the same order along the line on which they lie. C and C' lie on opposite sides of this line. Hence if AB be moved into coincidence with $A'B'$ the sides AC will not take the direction $A'C'$ but will lie on the other hemisphere from it. Consequently the angles are not congruent and must be symmetrical. In like manner the other corresponding angles may be proved to be symmetrical. The triangle $A'B'C'$ therefore satisfies the six conditions of congruent sides and symmetrical angles imposed in the definition given at the beginning of the lecture. The theorems concerning symmetrical triangles are valid and the development of such theorems as the following may be left to the reader with merely a caution

generally to substitute "symmetrical" for the "equal" which is found in most books or for the "congruent" found in some.

Theorem 58. *The bisector of the vertical angle of an isosceles triangle, the perpendicular bisector of the base, and the perpendicular let fall from the vertex to the base are all coincident.*

Theorem 59. *In a triangle symmetrical angles are opposite congruent sides and conversely. In case the angles are not symmetrical the greater angle is opposite the greater side and conversely.*

Theorem 60. *Of oblique lines drawn from a point to a line those cutting off the greater segments from the foot of the perpendicular are the greater or the less according as the perpendicular is less or greater than a quadrant.*

To quote the theorems concerning the locus of points equidistant from two given points or from two given lines and others like them would be merely tedious. Enough has been said to show the great importance of the ideas of symmetry to the development of spherical geometry and of plane geometry in case all operations are to be restricted to a plane. The really instructive part of the work was the necessity of establishing the existence of such objects as symmetrical triangles. It may be noted that as yet no construction for the middle point of a line or for the bisector of an angle or for a right angle has been given. These come in more appropriately under the topic of circles. The knowledge of the existence of the elements desired and not a method of finding them has been all that has proved necessary. It may also be noted that no measure either of segments or of angles has been introduced. The ideas of congruence, of "greater than" and "less than" which were derived from congruence, and even of the sum of two segments could all be carried out in a purely geometrical way by aid of Axiom V with no assistance from the theory of number.

A SIMPLE CONSTRUCTION FOR FINDING THE DIAMETER OF A GIVEN MATERIAL SPHERE.

By H. S. QUACKENBUSH, Lawrence Scientific School, Harvard University.

The following construction for finding the diameter of a given material sphere appears simpler and more direct than the one given in Wentworth's, Phillips and Fisher's, and other current text-books.

From any two points P and Q on the sphere as poles, with compass-opening a, describe two arcs on the sphere, intersecting at A and B. From the same two points as poles, with a different compass-opening b, describe two more arcs, intersecting at C and D. These four points, A, B, C, D, will then lie on a great circle of the sphere, namely, the perpendicular bisector of the arc PQ.

To find the diameter of the sphere we have then simply to choose any three of these four points, say A, B, C, measure with the compasses the straight line distances between them, lay off the plane triangle ABC, and find the diameter of its circumscribed circle.

This construction, like the one usually given, requires only a small part of the spherical surface to be accessible. A good check on the accuracy of the work can be obtained by making a different choice of the three points.

ON THE TOTITIVES OF DIFFERENT ORDERS.

By Professor G. A. MILLER, Stanford University.

While the close connection between number theory and group theory has been frequently observed yet this connection has been explicitly developed in comparatively few cases. Such developments are especially important from a pedagogical standpoint since much time can be saved by studying related subjects from a common point of view. The following note aims to exhibit an interesting application of group theory to the totitives of different orders.

Let $\varphi(m)$ represent the number of natural numbers which are both prime to m and do not exceed m. The formula

$$\varphi(m)=m(1-\frac{1}{p_1})(1-\frac{1}{p_2})\ldots\ldots\ldots(1-\frac{1}{p_n}),$$

$p_1, p_2, \ldots\ldots\ldots, p_n$ being the distinct primes which divide m, is known as Euler's φ formula. The value of $\varphi(m)$ is called the indicator of m, according to Cauchy. Sylvester has called it the totient of m, while the $\varphi(m)$ numbers themselves are called the totives of m. The formula has been generalized in a large number of ways.* Jordan,† Klein‡ and others have employed the following generalization.

Consider the number of different sets of k numbers such that none of these numbers exceeds m and that the greatest common divisor of m and all the numbers of any set is unity. The number of these sets is denoted by $\varphi_k(m)$ and is called the totient or indicator of order k with respect to m. When $k=1$ it reduces to the ordinary totient and the subscript is usually omitted. The value of $\varphi_k(m)$ may be determined as follows:

Suppose that an abelian group (G) has k independent generators $(s_1, s_2, \ldots\ldots\ldots, s_k)$ of order m. It is well known that every operator of G can be written in the form

$$s=s_1^{a_1}s_2^{a_2}\ldots\ldots\ldots s_k^{a_k}, \quad a_1, a_2, \ldots\ldots\ldots, a_k=1, 2, \ldots\ldots\ldots, m.$$

The order of s is m whenever the greatest common divisor of $m, a_1, a_2, \ldots\ldots\ldots, a_k$ is unity, and conversely. Hence $\varphi_k(m)$ is equal to the numbers of operators of order m in G. To find this number it is convenient to consider a different set of independent generators of G. If

$$m=p_1^{\beta_1}p_2^{\beta_2}\ldots\ldots\ldots p_n^{\beta_n},$$

G is the direct product of n subgroups $(H_1, H_2, \ldots\ldots\ldots, H_n)$ of orders $p_1^{\beta_1 k}$,

*Cf. *Encyclopaedie der Mathematischen Wissenschaften*, Vol. 1, p. 557; Hagen, *Synopsis der Hoeheren Mathematik*, Vol. 1, p. 12; Lucas, *Theorie des Nombres*, Vol. 1, p. 399.

†Jordan, *Traite des Substitutions*, p. 96.

‡Klein-Fricke, *Theorie der Elliptischen Modufunctionen*, Vol. 1, p. 397.

$p_2^{\beta_2 k}$, ..., $p_n^{\beta_n k}$, respectively. Each of these subgroups has k independent operators of orders $p_1^{\beta_1}$, $p_2^{\beta_2}$, ..., $p_n^{\beta_n}$, respectively. These independent generators are also independent generators of G.*

In order to obtain an operator of order m it is necessary and sufficient that its factors from H_1, H_2, ..., H_n, respectively, be of the highest possible orders; *i. e.*, these factors must be of orders $p_1^{\beta_1}$, $p_2^{\beta_2}$, ..., $p_n^{\beta_n}$. The number of operators of $p_1^{\beta_1}$ in H_1 is clearly equal to its order diminished by the order of a group having its k invariants equal to $p_1^{\beta_1-1}$. As similar remarks apply to the other subgroups we have

$$\varphi_k(m)=p_1^{(\beta_1-1)k}\, p_2^{(\beta_2-1)k} \cdots\cdots p_n^{(\beta_n-1)k}\,(p_1^k-1)(p_2^k-1)\cdots\cdots(p_n^k-1)$$

$$=m^k\left(1-\frac{1}{p_1^k}\right)\left(1-\frac{1}{p_2^k}\right)\cdots\cdots\left(1-\frac{1}{p_n^k}\right).$$

It is well known that the $\varphi(m)$ numbers which are the totitives of m form a congruence group with respect to multiplication, viz., the group of isomorphisms of the cyclic group of order m. The total number of totitives of any higher order than the first do not form a group. If we take only those sets in which each of the k elements is prime to m we obtain $[\varphi(m)]^k$ sets which clearly form a group when the corresponding elements of the sets are multiplied together. This is the direct product of k groups of isomorphisms of the cyclic group of order m.

The above method of finding the value of $\varphi_k(m)$ is not any simpler than the one which does not involve the theory of abelian groups.† The main object of this note is to call attention to the fact that this important function is involved in the determination of the number of operators of highest order of some special type of abelian groups, while it is not difficult to determine the number of operators of every order in any abelian group.‡

From the standpoint of group theory the given $[\varphi(m)]^k$ sets of k numbers which are separately prime to m furnish an interesting extension of the important system of congruence groups formed by the totitives of m.§ It should however be observed that it is not possible to represent all abelian groups by such sets of numbers. All abelian groups having k invariants may however be represented as additive groups whose operators are sets of k numbers taken separately with respect to k mnduli m_1, m_2, ..., m_k, which are respectively equal to the invariants. In fact this is ordinarily done when the operators are represented by an index system.

*As the order of each of these nk independent generators is a power of some prime number these orders constitute the maximum number of invariants of G and they are completely determined by this property.

†Cf. *Theorie des Nombres*, 1900, p. 36.

‡Cf. Netto, *Vorlesungen ueber Algebra*, Vol. 2, 1900, p. 247.

§Weber, *Lehrbuch der Algebra*, Vol. 2 (1899), p. 60; Pund, *Mitteilungen der Mathematischen Gesellschaft in Hamburg*, Vol. 3 (1899), p. 371.

AN ELEMENTARY EXPOSITION OF THE THEORY OF FINITE DIFFERENCES.*

By SAUL EPSTEEN.

§1.

In the Differential Calculus we study the *limit* of the ratio $[f(x+\Delta x)-f(x)] \div \Delta x$ as Δx approaches 0; in the Calculus of Finite Differences however we study it as a ratio in which Δx is finite.

The function of x under consideration being u, it is customary to write it u_x and to designate its increment by Δu_x where $\Delta u_x = u_{x+\Delta x} - u_x$. In the Differential Calculus $\frac{du}{dx}$ is not a fraction but a symbol of operation, du, dx having no meanings by themselves. In the Calculus of Finite Differences on the other hand $\frac{\Delta u_x}{\Delta x}$ is a true fraction.

Instead of considering $\frac{\Delta u_x}{\Delta x}$, the following simplification is ordinarily introduced. Letting $\Delta x = h$ and writing $x = ht$ it follows at once that for h=constant, $\Delta x = h \Delta t$, whence $\Delta t = 1$. Thus it is seen that by a simple transformation of the real variable x, the increment may be taken as unity; replacing t by x, $\Delta x = 1$ and the attention need now be directed merely to the operation Δ, defined by the equation $\Delta u_x = u_{x+1} - u_x$. The above simplification consists at bottom of selecting the finite increment Δx as the unit of measure.

§2.

Since Δu_x is itself a function of x its increment $\Delta(\Delta u_x)$ may be found by the same process as above; it is ordinarily designated by $\Delta^2 u_x$. In a like manner the differences $\Delta^2 u_x$, $\Delta^3 u_x$, ... may be found. As an example of this notion of successive differences consider the very simple function $u_x = x^2$. Evidently $\Delta u_x = u_{x+1} - u_x = 2x+1$; $\Delta^2 u_x = 2$. The following table may therefore be made:

value of x	1	2	3	4	5
u_x	1	4	9	16	25.....
Δu_x	3	5	7	19	.....
$\Delta^2 u_x$	2	2	2	.....	.....

It is interesting to observe that if any term be subtracted from the one immediately following to the right, the difference will be the term immediately below it. This is general of course, being due to the fact that $\Delta u_x = u_{x+1} - u_x$.

It will now be shown that if $u_x = ax^n$, $\Delta^n u_x = an!$

By definition, $\Delta u_x = a(x+1)^n - ax^n = ax^{n-1} + b_1 x^{n-2} + b_2 x^{n-3} + \ldots\ldots$, b_1, b_2,

*Literature: A Treatise on the Calculus of Finite Differences, by George Boole; Theorie der Differenzen und Summen, by O. Schloemilch; Differenzenrechnung, by A. A. Markoff; Differenzenrechnung, by D. Seliwanoff in the *Encyklopaedia der Mathematischen Wissenschaften*, Bd. 1, Heft 6.

....., being constants; repeating the process the result may be written $\Delta^2 u_x = an(n-1)x^{n-2} + c_1 x^{n-3} + c_2 x^{n-4} + \ldots\ldots$, $c_1, c_2, \ldots\ldots$ being constants; continuing in this manner the desired result is reached.

Evidently $\Delta^r ax^s = 0$ if $r > s$, hence

$$\Delta^n (a_1 x^n + a_2 x^{n-1} + \ldots\ldots + a_{n-1}x + a_n) = a_1 n!$$

An expression

$$x(x-1)(x-2)\ldots\ldots(x-m+1) \equiv x^{(m)}$$

is known as a *factorial.* It is important to observe that *in the Calculus of Finite Differences factorials play an analogous rôle to that of powers of x in the Differential Calculus.*

Thus, $\frac{dx^m}{dx} = mx^{m-1}$; now $\Delta x^{(m)} = (x+1)^{(m)} - x^{(m)}$

$$= (x+1)x(x-1)\ldots\ldots(x-m+2) - x(x-1)\ldots\ldots(x-m+1) = mx^{(m-1)}.$$

The following theorem of the Calculus of Finite Differences reminds one of Maclaurin's theorem in Differential Calculus: Let $\phi(x)$ be a rational integral function of x of the mth degree, then it may be expanded in a series of factorials thus:

$$(1)\quad \phi(x) = \phi(0) + \Delta(0).x + \frac{\Delta^2\phi(0)}{2!}x^{(2)} + \frac{\Delta^3\phi(0)}{3!}x^{(3)} + \ldots + \frac{\Delta^m\phi(0)}{m!}x^{(m)}.$$

In proof, assume

$$\phi(x) = a + bx + cx^{(2)} + dx^{(3)} + \ldots\ldots + hx^{(m)},$$

then

$$\Delta\phi(x) = b + 2cx + 3dx^{(2)} + \ldots\ldots + mhx^{(m-1)},$$

$$\Delta^2\phi(x) = 2c + 3.2dx + \ldots\ldots + m(m-1)hx^{(m-2)},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Delta^m\phi(x) = m(m-1)\ldots\ldots 2.1.h.$$

From this it is clear that

$$a = \phi(0),\ b = \Delta\phi(0),\ c = \frac{\Delta^2\phi(0)}{2!},\ \ldots\ldots,\ h = \frac{\Delta^m\phi(0)}{m!}.$$

§3.

In §6 difference equations will be studied of the general form

$$(2)\qquad \varphi_1(x, u_x, u_{x+1}, \ldots\ldots, u_{x+n}) = 0.$$

The considerations of §§1, 2 would lead one to suppose however that a linear difference equation should have the form

(3) $$\varphi_2(x, \Delta u_x, \Delta^2 u_x, \ldots\ldots, \Delta^n u_x)=0$$

where $$\Delta^2 u_x=\Delta(\Delta u_x),\ \Delta^3 u_x=\Delta(\Delta^2 u_x),\ \ldots\ldots$$

The equations (2) and (3) are in reality equivalent. First, by demonstrating that for any integer j the difference $\Delta^j u_x$ can be expressed linearly with constant coefficients in terms of $u_{x+j}, u_{x+j-1}, \ldots\ldots, u_{x+1}, u_x$ it will be concluded that an equation of the form (3) may be replaced by one of the form (2); secondly, by showing that u_{x+j} can be expressed linearly with constant coefficients in terms of $\Delta^j u_x, \Delta^{j-1} u_x, \ldots\ldots, \Delta u_x, u_x$ it will be concluded that an equation of the form (2) can be replaced by one of the form (3).

I. By definition $\Delta u_x=u_{x+1}-u_x$, whence

$$\Delta^2 u_x=\Delta(\Delta u_x)=(u_{x+2}-u_{x+1})-(u_{x+1}-u_x)=u_{x+2}-2u_{x+1}+u_x.$$

By the ordinary method of induction it may be seen that

(4) $$\Delta^j u_x=u_{x+j}-ju_{x+j-1}+\frac{j(j-1)}{1.2}u_{x+j-2}+\ldots\ldots+(-1)^j u_x.$$

II. From $\Delta u_x=u_{x+1}-u_x$ it is seen that $u_{x+1}=u_x+\Delta u_x$, whence

$$u_{x+2}=u_x+\Delta u_x+\Delta(u_x+\Delta u_x)=u_x+2\Delta u_x+\Delta^2 u_x.$$

It follows now by induction that

(5) $$u_{x+j}=u_x+j\Delta u_x+\frac{j(j-1)}{1.2}\Delta^2 u_x+\ldots\ldots+\Delta^j u_x.$$

§4. Applications.

I. Writing 0 for x and x for j in equation (5) there results

(5′) $$u_x=u_0+x\Delta u_0+\frac{x(x-1)}{2!}\Delta^2 u_0+\ldots\ldots.$$

Suppose now there is given log3.14=.4969296, log3.15=.4983106, log3.16 =.4996871, log3.17=.5010593; required an approximate value of log3.14159.

The following table with the decimal point omitted may be written down at once:

	u_0	u_1	u_2	u_3
	4969296	4983106	4996871	5010593
Δ	13810	13765	13722	
Δ^2	−45	−43		
Δ^3	2.			

The common difference of 3.14, 3.15, 3.16, 3.17 is .01; this being taken as the unit gives the value of x which corresponds to 3.14159 as .159. Hence by (5′),

$$(6)\quad u_x=4969296+.159\times 13810+\tfrac{1}{2}(.159)(.159-1)\times(-45)$$
$$+\tfrac{1}{6}(.159)(.159-1)(.159-2)\times 2;$$

whence $u_x=.4971495$, which is correct to the seventh decimal place.

II. In Life Insurance it is customary to denote the number of men living at an age x by l_x. Suppose there is given a Mortality Table, obtained from statistics, containing the values of l for intervals of five years; the problem is to interpolate for the intermediary values. The formula (5′) is applied.

Let u_x be the desired value of the number living at the age x, $l_{x\pm z}=l'_z$ the number living at the nearest five-year period which is z years younger or older than the age at which the value is desired. Evidently $z\not>2$.

If $u_0=l'_{z-5}$, $u_1=l'_z$, $u_2=l'_{z+5}$, $u_x=l_x$, it is seen that

$$\Delta u_0=l'_z-l'_{z-5},\ \Delta u_1=l'_{z+5}-l'_z, \text{ and}$$
$$\Delta^2 u_0=(l'_{z+5}-l'_z)-(l'_z-l'_{z-5})=l'_{z+5}-2l'_z+l'_{z-5}.$$

Inserting in (5′) and stopping at the second differences,

$$(7)\qquad l_x=l'_{z-5}+x(l'_z-l'_{z-5})+\frac{x(x-1)}{2}(l'_{z+5}-2l'_z+l'_{z-5}).$$

It should be kept in mind that in (7) z is expressed in units of one year while x is expressed in units of five years.

III. The area bounded by a curve, two ordinates u_0, u_2 and the axis of x being divided into two equal parts by the ordinate u_1, an approximate value for the area is required, the distance between u_0 and u_1 being taken as the unit of length.

The general ordinate being u_x, the expression for the required area $\int_0^2 u_x dx$ becomes, by means of (5′),

$$\int_0^2 u_x dx=u_0\int_0^2 dx+\Delta u_0\int_0^x x dx+\frac{\Delta^2 u_0}{2}\int_0^2 x(x-1)dx=2u_0+2\Delta u_0+\tfrac{1}{3}\Delta^2 u_0.$$

Since $\Delta u_0=u_1-u_0$, $\Delta^2 u_0=u_2-2u_1+u_0$, it follows that

$$\int_0^2 u_x dx=\frac{u_0+4u_1+u_2}{3}.$$

This is known in the Calculus as Simpson's Rule, the generalization of which for the numeral solution of differential equations was made by Professor C. Runge in the *Mathematische Annalen*, volume 46.

Consider the interval u_0u_6 divided into six parts by the seven equidistant ordinates $u_0, u_1, u_2, \ldots\ldots\ldots, u_6$, the distance between the two adjacent ordinates being taken as the unit of length. The formula for the area becomes by (5′):

$$\int_0^6 u_x dx=6u_0+18\Delta u_0+27\Delta^2 u_0+24\Delta^3 u_0+\tfrac{123}{10}\Delta^4 u_0+\tfrac{33}{10}\Delta^5 u_0+\tfrac{41}{140}\Delta^6 u_0.$$

The last coefficient, $\frac{41}{140}$, differs from $\frac{42}{140}$ or $\frac{3}{10}$ by the small fraction $\frac{1}{140}$, and as from the nature of the approximation the sixth differences are supposed to be small, the error will be very slight if the last term be changed to $\frac{3}{10}\triangle^6 u_0$. By means of §2, I, it follows now that:

$$\int_0^6 u_x dx = \tfrac{3}{10}[u_0+u_2+u_4+u_6+5(u_1+u_3+u_5)+u_3].$$

This very elegant approximation for the area reads in words: Divide the area into six equal parts by means of seven ordinates and take three-tenths of the sum of the odd ordinates plus five times the sum of the even ordinates plus the middle ordinate; the distance between the ordinates being the unit of length.

The generalization of this interesting rule for the numerical solution of differential equations has not yet been made as far as I am aware.

§5.

The term "finite integration" is employed to denote a process whose relation to $\triangle$ is analogous to the relation of integration to $D_x=\frac{d}{dx}$.

In the Calculus integration may be defined in two ways, first as an area $\int_a^x f(x)dx$ and secondly as the inverse of differentiation, *i. e.*, $D_x^{-1}f(x)$, the former being related to the latter by the equation

$$\int_a^x f(x)dx=\Big[D_x^{-1}f(x)\Big]_a^x.$$

The situation in the theory of finite differences is similar; the finite integral of u_x may be also defined in two ways.

I. Let $\triangle v_x=u_x$, let a be some constant and take u_a as the initial term, then $v_x=S_a^x u_x{}^*=u_a+u_{a+1}+\ldots\ldots\ldots+u_{x-1}$.

For $v_{x+1}=u_a+u_{a+1}+\ldots\ldots\ldots+u_{x-1}+u_x$, whence $v_{x+1}-v_x=\triangle v_x=u_x$.

II. $Su_x=[\triangle^{-1}u_x]$. The number of expressions satisfying this second definition is infinite since it includes $u_a+u_{a+1}+\ldots\ldots\ldots+u_{x+1}$ for all values of a. But just as in the Integral Calculus it may be shown that $S_a^x u_x=[\triangle^{-1}u_x]_a^x$.

Let $f(x)$ be a function such that $\triangle f(x)=u_x$,

$$\begin{aligned} f(a+1)-f(a)&=u_a\\ f(a+2)-f(a+1)&=u_{a+1}\\ &\cdots\\ f(x)-f(x-1)&=u_{x-1},\end{aligned}$$

hence, by addition, $f(x)-f(a)=S_a^x u_x$,

and from $\triangle f(x)=u_x$, which gives $f(x)=\triangle^{-1}u_x$, it is seen that $S_a^x u_x=[\triangle^{-1}u_x]_a^x$,

*We regard the use of S in place of the customary Σ as preferable, first because it corresponds closer to the $\int$ of the Integral Calculus; secondly, because it leaves the Σ open for use in denoting ordinary summations; and thirdly, when the integration cannot be performed and the S must be regarded as a functional symbol, there cannot be any confusion as may possibly occur in the case of the Σ.

the value for the lower limit being subtracted from the value for the upper limit. The above makes it possible to speak of definite and indefinite finite integrals.

III. Let $Sv_x=f(x)$ be defined according to I and let $\triangle^{-1}u_x=f(x)+w_x$; then $u_x=\triangle f(x)+\triangle w_x=u_x+\triangle w_x$, whence $\triangle w_x=0$; that is to say, $w=$constant, and therefore $\triangle^{-1}u_x=Sv_x+C$.

IV. In §3 it was shown that $\triangle x^{(m)}=mx^{(m-1)}$, where $x^{(m)}=x(x-1)\ldots\ldots(x-m+1)$. This gives immediately an illustration of finite integration, namely: $Sx^{(m)}=\dfrac{x^{(m+1)}}{m+1}+C$. By substituting $u_x=Sv_x$ in (5'), $u_x=u_0+x\triangle u_0+\dfrac{x^{(2)}\triangle^2 u_0}{2!}+\ldots\ldots$ and writing $c=u_0$, there results

$$Sv_x=cx+v_0+\frac{x^{(2)}}{2!}\triangle v_0+\ldots\ldots \tag{8}$$

the number of terms being finite if v_x is rational and integral.

Suppose now that it be required to find the sum of $1^2+2^2+\ldots\ldots+x^2$. Applying (8) and observing that $\triangle v_0=1$, $\triangle^2 v_0=2$,

$$1^2+2^2+\ldots\ldots+(x-1)^2=Sx^2=cx+\frac{x(x-1)}{2}+\frac{x(x-1)(x-2)}{3!}.2.$$

When $x=2$, $c=0$, and adding x^2 to both sides it follows by easy reductions that $1^2+2^2+\ldots\ldots+x^2=\frac{1}{6}x(x+1)(2x+1)$.

§6.

It sometimes happens that instead of the difference $\triangle u$ of the function u being given to determine u, some relation between its successive differences is given. As illustration, let $\triangle^2 u_x=\dfrac{2}{x+1}\triangle u_x-\dfrac{2}{x(x+1)}\triangle x$; it may easily be verified that the solution is $u=ax^2+bx$.

In general, if the relation $f(x, u_x, c_1, \ldots\ldots, c_n)=0$ is given, and if $c_1, \ldots\ldots, c_n$ are eliminated by means of the successive differences $\triangle^i f(x, u_x, c_1, \ldots\ldots, c_n)=f_i(x, u_x, c_1, \ldots\ldots, c_n)=0$ $(i=1\ldots\ldots n)$, there will result a *difference equation*, $F(x, u_x, \triangle u_x, \ldots\ldots, \triangle^n u_x)=0$, whose *complete primitive* $f(x, u_x, c_1, \ldots\ldots, c_n)=0$ contains n arbitrary constants. By §2, I, the latter equation can be replaced by one in $u_x, u_{x+1}, \ldots\ldots, u_{x+n}$; solved for u_{x+n} it has the form

$$u_{x+n}=\psi(x, u_x, u_{x-1}, \ldots\ldots, u_{x+n-1}). \tag{9}$$

It will be observed that the independent variable x takes on values which differ by unity, hence $u_{x+n+1}=\psi(x+1, u_{x+1}, \ldots\ldots, u_{x+n})$; eliminating u_{x+n} by means of (9) the result may be written $u_{x+n+1}=\psi(x, u_x, \ldots, u_{x+n-1})$. Similarly

$$u_{x+n+m}=\psi_m(x, u_x, \ldots\ldots, u_{x+n-1}) \quad (m=1, 2, 3, \ldots\ldots). \tag{10}$$

It is now clear that the difference equation (9) is a recursion formula for the function u in terms of the n arbitrary values at the points x, $x+1$, $\ldots\ldots$, $x+n-1$. Replacing $x+n+m$ by x the general expression for the function is obtained. It should be noticed however that it is tacitly assumed that the function ψ is finite at the points x, $x+1$, $\ldots\ldots$, $x+n+m$. Thus, if the equation is linear

$$u_{x+n}+p_x^{(1)}u_{x+n-1}+\ldots\ldots+p_x^{(n)}u_x=0,$$

the above discussion applies only when the coefficients $p_x^{(1)}, \ldots\ldots, p_x^{(n)}$ are finite at the points x, $x+1$, $x+2$, $\ldots\ldots$ The general linear homogeneous equation of the first order may be written in either of the equivalent forms $\triangle u_x+q_x u_x=0$, $u_{x+1}=p_x u_x$. Starting with $x=a$ and taking successive values,

$$u_{a+1}=p(a)u_a, \quad u_{a+2}=p(a+1)u_{a+1}, \ldots\ldots, u_{a+m}=p(a+m-1)u_{a+m+1},$$

this gives at once $u_{a+m}=p(a).p(a+1), \ldots\ldots, p(a+m-1)u_a$. Writing $a+m=x$ $u_a=c$, we have $u_x=c.p(x-1).p(x-2).\ldots\ldots.p(a+1).p(a)$ which is the general solution sought.

Just as in the theory of linear differential equations certain other special cases, such as equations with constant coefficients, can be integrated. For the linear equation of the nth order $(n>1)$ with variable coefficients no general methods are known.

CHICAGO, *June, 1904.*

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

198. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Solve $2^{x+y}=6^y$; $3^x=3.2^{y+1}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and A. H. HOLMES, Brunswick, Me.

$2^{x+y}=6^y$, gives $2^x=3^y$.....(1), $3^x=3.2^{y+1}$, gives $3^{x-1}=2^{y+1}$.....(2).

(1) multiplied by (2) gives $6^x=6^{y+1}$.....(3).

From (1), $x/y=\log 3/\log 2$; from (3), $x=y+1$.

$\therefore y=\log 2/(\log 3-\log 2)$, $x=\log 3/(\log 3-\log 2)$.

Also solved by F. D. Posey, J. E. Sanders, G. W. Greenwood, Christian Hornung, L. E. Newcomb, J. Scheffer, Grace M. Bareis, and the Proposer.

199. Proposed by SAUL EPSTEEN, Ph. D., Chicago, Ill.

Solve $(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_6)=(x+a_1)(x+a_2)(x+a_3)(x+a_4)(x+a_5)(x+a_6)$.

Solution by F. D. POSEY, A. B., San Mateo, Cal., and G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $p_1=\Sigma a_1$, $p_2=\Sigma a_1 a_2$, etc. Expanding both members of the equation we have $x^6-p_1x^5+p_2x^4-p_3x^3+p_4x^2-p_5x+p_6=x^6+p_1x^5+p_2x^4+p_3x^3+p_4x^2+p_5x+p_6$, which reduces to $p_1x^5+p_3x^3+p_5x=0$. One root is therefore 0. Dividing by x, we have $p_1x^4+p_3x^2+p_5=0$.

Solving the latter as a quadratic we have $x^2=[-p_3\pm(p_3^2-4p_1p_5)^{\frac{1}{2}}]/2p_1$.

$$\therefore x=\pm\left[\frac{-p_3\pm(p_3^2-4p_1p_5)^{\frac{1}{2}}}{2p_1}\right]^{\frac{1}{2}}.$$

Since the original equation is of the sixth degree, one root is infinite.

Also solved by G. W. Greenwood, B. A. (Oxon), and L. E. Newcomb.

GEOMETRY.

220. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.

Two triangles are circumscribed to a given triangle ABC, having their sides perpendicular to the sides of the given triangle. Prove that the two triangles are equal, and find the area of these triangles.

III. Solution by J. SCHEFFER, Kee Mar College, Hagerstown, Md.

Let $A'B'=c'$ be the side of the circumscribed triangle drawn through B and perpendicular to AB, $A'C'=b'$ through A and perpendicular to AC, and $B'C'=a'$ through C and perpendicular to BC. It is seen at once that triangle $A'B'C'$ is equiangular with triangle ABC. We have

$$A'B'=c'=c\cot A+\frac{a}{\sin B}=\frac{c\cos A}{\sin A}+\frac{c\sin A}{\sin B\sin C}=c\,\frac{\cos A\sin B\sin C+\sin^2 A}{\sin A\sin B\sin C}$$

$$=c\,\frac{1-\cos A(\cos A-\sin B\sin C)}{\sin A\sin B\sin C}=c\,\frac{1+\cos A\cos B\cos C}{\sin A\sin B\sin C}$$

$$=c(\operatorname{cosec}A\operatorname{cosec}B\operatorname{cosec}C+\cot A\cot B\cot C)=c(\cot A+\cot B+\cot C).$$

By analogy $b'=b(\cot A+\cot B+\cot C)$, $a'=a(\cot A+\cot B+\cot C)$. This proves the equality of all circumscribed triangles, the ratio $(\cot A+\cot B+\cot C)$ being the same. Denoting the triangle ABC by $\triangle$, this ratio may also be expressed $\frac{a^2+b^2+c^2}{4\triangle}$. Denoting the area of the circumscribed triangle by $\triangle'$, we have evidently $\triangle'=(\cot A+\cot B+\cot C)^2\triangle=\frac{(a^2+b^2+c^2)^2}{16\triangle}$.

222. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

At the ends of a focal chord CC' of a parabola are drawn the normal chords CD, $C'D'$. Prove that DD' is parallel to CC' and equal to three times its length.

III. Solution by J. SCHEFFER, Kee Mar College, Hagerstown, Md.

It is a well-known theorem that the two tangents at the extremities of any focal chord in a parabola intersect at right angles on the directrix. Choosing two such tangents for the axes of coördinates, the lengths of the tangents between the origin and points of contact being a and b, we have $(x/a+y/b-1)^2=4xy/ab$ as the equation of the parabola, and $x/a+y/b=1$ the equation of the focal chord. The equation of the normal at C' is $x=a$, and at C, $y=b$. Substituting these in the equation of the parabola we get for the point D, $(a, 4b)$ and for D', $(4a, b)$. Consequently the equation for DD' is $y=-(b/a)x+3b$, which shows that it is parallel to CC'. $DD'=\sqrt{(9a^2+9b^2)}=3\sqrt{(a^2+b^2)}$; and $CC'=\sqrt{(a^2+b^2)}$, which proves $DD'=3CC'$.

223. Also solved by L. E. Newcomb, Los Gatos, Cal.

224. Proposed by WILLIAM HOOVER, Ph. D., Professor of Mathematics, Ohio State University, Athens, O.

The equations to two circles are $(x-a)^2+(y-b)^2=c^2$, $(x-b)^2+(y-a)^2=c^2$; give the length of their common tangent and *thence* the condition that the two circles may touch.

Solution by J. SCHEFFER, Hagerstown, Md.; F. D. POSEY, A. B., San Mateo, Cal.; G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.; A. H. HOLMES, Brunswick, Me.; J. E. SANDERS, Hackney, O.; CHRISTIAN HORNUNG, A. M., Tiffin, O.; E. L. SHERWOOD, Pittsburg, Pa.; GRACE M. BAREIS, Bala, Pa.

Since the circles are equal the length of a common external tangent is equal to the distance between the centers which is $\sqrt{[(a-b)^2+(b-a)^2]}=(a-b)\sqrt{2}$.

$\therefore$ The circles will touch if $(a-b)^2=2c^2$.

Also solved by L. E. Newcomb, Los Gatos, Cal.

225. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Determine the sides of a triangle, given the lengths of (1) the three altitudes, (2) the three medians, (3) the radii of the escribed circles, (4) the radius R of the circumscribed circle and any two of the three quantities r=radius of inscribed circle, s=semi-perimeter, $\triangle$=area.

Solution by the PROPOSER.

(1) Let A be the altitude on side a, B on side b, C on side c.

$\therefore \triangle = \sqrt{[s(s-a)(s-b)(s-c)]} = \frac{1}{2}aA = \frac{1}{2}bB = \frac{1}{2}cC$,

$b = a(A/B)$, $c = a(A/C)$, $2s = a+b+c = a(1+A/B+A/C)$,

$2(s-a) = a(-1+A/B+A/C)$, $2(s-b) = a(1-A/B+A/C)$,

$2(s-c) = a(1+A/B-A/C)$.

$\therefore a = 2/AW, b = 2/BW, c = 2/CW$, $W = \sqrt{[(1/A+1/B+1/C)(-1/A+1/B+1/C)(1/A-1/B+1/C)(1/A+1/B-1/C)]}$.

(2) Let A be the median to side a, etc.; O the intersection of the medians. From the triangle with base a and vertex O, we have

$$(\tfrac{2}{3}C)^2 + (\tfrac{2}{3}B)^2 = 2(\tfrac{1}{3}A)^2 + 2(\tfrac{1}{2}a)^2, \text{ whence } a^2 = \tfrac{8}{9}B^2 + \tfrac{8}{9}C^2 - \tfrac{4}{9}A^2.$$

Similarly, $b^2 = \frac{8}{9}A^2 + \frac{8}{9}C^2 - \frac{4}{9}B^2$, $c^2 = \frac{8}{9}A^2 + \frac{8}{9}B^2 - \frac{4}{9}C^2$.

(3) With the usual notation for the radii of the escribed circles,

$$r_1 = \frac{\triangle}{s-a}, \quad r_2 = \frac{\triangle}{s-b}, \quad r_3 = \frac{\triangle}{s-c}, \quad \triangle^2 = rr_1r_2r_3, \quad \triangle = rs.$$

$$\therefore b+c-a = 2s-2a = 2\triangle/r_1, \quad a-b+c = 2\triangle/r_2, \quad a+b-c = 2\triangle/r_3.$$

$$\therefore a = \triangle(1/r_2+1/r_3), \quad b = \triangle(1/r_1+1/r_3), \quad c = \triangle(1/r_1+1/r_2).$$

$$\therefore s = \triangle(1/r_1+1/r_2+1/r_3), \quad 1/r = 1/r_1+1/r_2+1/r_3),$$

$$\triangle^2 = rr_1r_2r_3 = \frac{r_1r_2r_3}{(1/r_1+1/r_2+1/r_3)}, \quad \triangle = \frac{r_1r_2r_3}{\sqrt{(r_1r_2+r_1r_3+r_2r_3)}}.$$

$$\therefore a = \frac{r_1(r_2+r_3)}{\sqrt{(r_1r_2+r_1r_3+r_2r_3)}}, \quad b = \frac{r_2(r_1+r_3)}{\sqrt{(r_1r_2+r_1r_3+r_2r_3)}}, \quad c = \frac{r_3(r_1+r_2)}{\sqrt{(r_1r_2+r_1r_3+r_2r_3)}}$$

(4) The three cases are equivalent since $\triangle = rs$. Let then R, r, s be given. The problem may be reduced to (3) since r_1, r_2, r_3 are the roots of

$$x^3 - (r+4R)x^2 + s^2x - rs^2 = 0.$$

To determine a, b, c directly, we employ $a+b+c = 2s$, $abc = 4\triangle R = 4rsR$, $(s-a)(s-b)(s-c) = \triangle^2/s = r^2s$.

$\therefore s^3 - s^2(a+b+c) + s(ab+ac+bc) - abc = r^2s$, $\quad ab+ac+bc = s^2+r^2+4rR$.

Hence a, b, c are the roots of $y^3 - 2sy^2 + (s^2+r^2+4rR)y - 4rsR = 0$.

Also solved by J. E. Sanders, J. Scheffer, and G. B. M. Zerr.

CALCULUS.

178. Proposed by SAUL EPSTEEN, Ph. D., The University of Chicago.

Evaluate $\int_0^{\frac{1}{2}\pi} \frac{d\phi}{1+\sin^2\phi}$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\int_0^{\frac{1}{2}\pi} \frac{d\varphi}{1+\sin^2\varphi} = \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{\sin^2\varphi+\cos^2\varphi+\sin^2\varphi} = \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{\cos^2\varphi+2\sin^2\varphi}$$

$$= \int_0^{\frac{1}{2}\pi} \frac{\sec^2\varphi d\varphi}{1+2\tan^2\varphi} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\sqrt{2}\tan\varphi\right)_0^{\frac{1}{2}\pi} = \frac{\pi}{2\sqrt{2}}.$$

II. Solution by A. H. HOLMES, Brunswick, Me.

Putting $\sin\phi=\tan\theta$, then $I=\int \frac{d\phi}{1+\sin^2\varphi} = \int \frac{\cos\theta d\theta}{\sqrt{(1-2\sin^2\theta)}}$

$$= \frac{1}{\sqrt{2}}\sin^{-1}(\sqrt{2}\sin\theta) = \frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}\sin\theta}{\sqrt{(1+\sin^2\theta)}}.$$

Hence, $\left(I\right)_0^{\frac{1}{2}\pi} = \frac{1}{\sqrt{2}}\sin^{-1}1 = \frac{\pi}{4}\sqrt{2}$.

III. Solution by CHRISTIAN HORNUNG, A. M., Heidelberg University, Tiffin, O., and GRACE M. BAREIS, Bala, Pa.

$$\int \frac{d\phi}{1+\sin^2\phi} = \int \frac{\sec^2\phi d\phi}{\sec^2\phi+\tan^2\phi} = \int \frac{\sec^2\phi d\phi}{1+2\tan^2\phi}.$$

If $\tan\phi=x$, $\int \frac{\sec^2\phi d\phi}{1+2\tan^2\phi} = \int \frac{dx}{1+2x^2} = \frac{1}{2}\int \frac{dx}{\frac{1}{2}+x^2} = \frac{1}{2}\frac{1}{\sqrt{\frac{1}{2}}}\tan^{-1}\frac{x}{\sqrt{\frac{1}{2}}}$

$= \frac{1}{2}\sqrt{2}\tan^{-1}x\sqrt{2}$, and, replacing the value of x, we have

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{1+\sin^2\phi} = \left(\frac{1}{2}\sqrt{2}\tan^{-1}\sqrt{2}\tan\phi\right)_0^{\frac{1}{2}\pi} = \frac{\pi\sqrt{2}}{4}.$$

Also solved by John G. Kellar, F. D. Posey, J. E. Sanders, J. Scheffer, S. A. Corey, G. W. Greenwood, E. L. Sherwood, Homer R. Higley, and L. E. Newcomb.

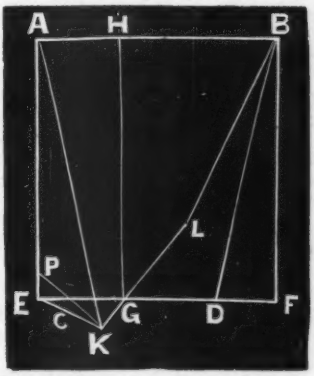
MECHANICS.

168. Proposed by M. E. GRABER, A. B., Instructor in Mathematics and Physics in Heidelberg University, Tiffin, Ohio.

In Bifilar Suspension W is the weight of the suspended mass, a and b the distances between the threads above and below, h the vertical height of the threads. If the difference in vertical components of tension is n times W and θ is the angle turned through in azimuth, momental resistance is $\frac{1}{4}(1-n^2)W(ab/h)\sin\theta$. [Perry's *Engineering*.]

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $AB=a$, $CD=b=KL$, $AE=HG=BF=h$, $\angle CGK=\angle LGD=\theta$, $AC=AK=l$, $BD=BL=l'$. Let T, T_1 be the vertical tensions at C and D, $CG=KG=x$, $GD=GL=y$. Then $T+T_1=W$, $T-T_1=nW$.



$\therefore T=\frac{1}{2}W(1+n)$, $T_1=\frac{1}{2}W(1-n)$.

$\therefore \frac{1}{2}W(1+n)x=\frac{1}{2}W(1-n)y$ and $x+y=b$.

$\therefore x=\frac{1}{2}b(1-n)$, $y=\frac{1}{2}b(1+n)$.

Similarly, $AH=EG=\frac{1}{2}a(1-n)$,

$HB=FG=\frac{1}{2}a(1+n)$.

Let $EAK=\phi$, $LBF=\psi$.

Q=the momental resistance at K.

Q'=the momental resistence at L.

Draw KP perpendicular to AE. Then $AP=l\cos\phi$.

$PK=l\sin\phi=\sqrt{(EG^2+KG^2-2EG.KG\cos\theta)}$.

$\therefore l\sin\phi=\frac{1}{2}(1-n)\sqrt{(a^2+b^2-2ab\cos\theta)}$.....(1).

Also $Qd\theta=-\frac{1}{2}W(1+n)d(l\cos\phi)=\frac{1}{2}W(1+n)l\sin\phi d\phi$.

$\therefore Qd\theta=\frac{1}{2}W(1+n)l\sin\phi d\phi$.....(2).

From (1), $l\cos\phi d\phi=\dfrac{ab(1-n)\sin\theta d\theta}{2\sqrt{(a^2+b^2-2ab\cos\theta)}}$.....(3).

Multiplying (1) by (3), we get

$$l^2\sin\phi\cos\phi\, d\phi=\tfrac{1}{4}ab(1-n)^2\sin\theta\, d\theta.....(4).$$

Eliminating $d\theta$, $d\phi$ between (2) and (4) we get

$$Q=\tfrac{1}{8}(1-n^2)(1+n)W(ab/l\cos\phi)\sin\theta.$$

Similarly, $$Q'=\tfrac{1}{8}(1-n^2)(1+n)W(ab/l'\cos\psi)\sin\theta.$$

Since the deflection is small we can place $l\cos\phi=l'\cos\psi=h$.

$\therefore Q=\frac{1}{8}(1-n^2)(1-n)W(ab/h)\sin\theta$, $Q'=\frac{1}{8}(1-n^2)(1+n)W(ab/h)\sin\theta$.

Momental resistance$=Q+Q'$.

$\therefore Q+Q'=\frac{1}{8}(1-n^2)(1-n+1+n)W(ab/h)\sin\theta=\frac{1}{4}(1-n^2)W(ab/h)\sin\theta$.

AVERAGE AND PROBABILITY.

152. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

A square hole is cut through the center of a sphere of radius r. Show that the average volume removed is $\frac{1}{12}r^3(23\sqrt{2}-28)$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $x^2+y^2+z^2=r^2$ be the equation to the sphere, $2b$=side of square hole.

Volume removed$=16\int_0^{\frac{1}{4}\pi}\int_0^{b\sec\theta} z\rho d\theta d\rho$. $z=\sqrt{(r^2-\rho^2)}$.

The limits of b are 0 and $\frac{1}{2}r\sqrt{2}=b'$. Let average volume$=\triangle$.

$$\therefore \triangle = 16\int_0^{\frac{1}{2}\pi}\int_0^{b}\int_0^{b\sec\theta} \sqrt{(r^2-\rho^2)}\,\rho\,d\theta\,db\,d\rho \Big/ \int_0^{b} db$$

$$= \frac{16\sqrt{2}}{3r}\int_0^{\frac{1}{2}\pi}\int_0^{b} [r^3-(r^2-b^2\sec^2\theta)^{\frac{3}{2}}]\,d\theta\,db$$

$$= \frac{r^3\sqrt{2}}{3}\int_0^{\frac{1}{2}\pi}\left[8\sqrt{2}-(5-\sec^2\theta)\sqrt{(2-\sec^2\theta)}-6\cos\theta\sin^{-1}\left(\frac{\sec\theta}{\sqrt{2}}\right)\right]d\theta$$

$$= \frac{r^3\sqrt{2}}{3}\left[2\pi\sqrt{2}-\tfrac{3}{2}\pi\sqrt{2}-\int_0^{\frac{1}{2}\pi}\left((5-\sec^2\theta)\sqrt{(2-\sec^2\theta)}-\frac{6\tan^2\theta}{\sqrt{(2-\sec^2\theta)}}\right)d\theta\right].$$

Let $\tan\theta=\sin\phi$.

$$\therefore \triangle = \frac{r^3\sqrt{2}}{3}\left[\tfrac{1}{2}\pi\sqrt{2}-\int_0^{\frac{1}{2}\pi}\frac{4\cos^2\phi-\sin^2\phi\cos^2\phi-6\sin^2\phi}{1+\sin^2\phi}\,d\phi\right]$$

$$= \frac{r^3\sqrt{2}}{3}\left[\tfrac{1}{2}\pi\sqrt{2}-\int_0^{\frac{1}{2}\pi}\left(\sin^2\phi-12+\frac{16}{1+\sin^2\phi}\right)d\phi\right] = \frac{\pi r^3}{12}(23\sqrt{2}-28).$$

Also solved by the Proposer.

GROUP THEORY.

2. Proposed by W. BURNSIDE, The Croft, Bromley Road, Catford, England.

Show that the group of the biquadratic equation $x^4+2ax^2+b=0$, in which a and b are rational numbers, while a^2-b is not the square of a rational number, is in general a dihedron group of order 8; but that (*i*) if b is the square of a rational number the group is a non-cyclical Abelian group of order 4; and (*ii*) if $b=a^2\div(1+n^2)$, where n is a rational number, the group is a cyclical group of order 4.

Solution by L. E. DICKSON, Ph. D., The University of Chicago.

The problem may be solved most readily by making use of the simple expressions resulting for the roots of this special quartic. To obtain a more instructive example of the application of groups to equations, I treat the problem without assuming the quartic to be solvable by radicals.

Consider the general quartic $x^4+ax^3+bx^2+cx+d=0$, and call its roots x_1, x_2, x_3, x_4. As is well known, the functions

$$y_1=x_1x_2+x_3x_4, \quad y_2=x_1x_3+x_2x_4, \quad y_3=x_1x_4+x_2x_3$$

are the roots of the cubic resolvent

$$y^3-by^2+(ac-4d)y-a^2d+4bd-c^2=0.$$

For our special quartic, this resolvent becomes

$$y^3-2ay^2-4by+8ab\equiv(y-2a)(y^2-4b)=0.$$

We may therefore, by fixing the notation of the x's, set $y_1=2a$, $y_2=2\sqrt{b}$, $y_3=-2\sqrt{b}$. These three values are distinct since a^2-b is not the square of a rational number by hypothesis. Further, y_1 has a rational value. Hence* the group G of the quartic is contained in the dihedron group G_8,

$$I,\ (12),\ (34),\ (12)(34),\ (13)(24),\ (14)(23),\ (1324),\ (1423),$$

which leaves $x_1x_2+x_3x_4$ formally unaltered. The only subgroups of order 4 of G_8 are the cyclic C_4, generated by (1324), and

$$G_4=[I,\ (12),\ (34),\ (12)(34)],\quad D_4=[I,\ (12)(34),\ (13)(24),\ (14)(23)].$$

Now the desired group G is contained in D_4 if and only if b is the square of a rational number. Indeed, $x_1x_3+x_2x_4=2\sqrt{b}$, from above, and $x_1x_3.x_2x_4=b$, from the quartic, so that $x_1x_3=x_2x_4=\sqrt{b}$. Since x_1x_3 is thus numerically unaltered by the substitions of D_4, it follows that G does not lie in D_4 if $\sqrt{b}$ is irrational. That G lies in D_4 if $\sqrt{b}$ is rational follows since the rational relation $x_1x_3=\sqrt{b}$ does not remain true upon applying (12), (34), (1324), or (1423), the quartic having no double root.

Next, G is not contained in G_4, which leaves x_1x_2 formally invariant. Indeed, $x_1x_2+x_3x_4=2a$ and $x_1x_2.\ x_3x_4=b$ give $x_1x_2=a\pm\sqrt{(a^2-b)}$, which is irrational.

Finally, G is contained in C_4 if and only if $(a^2-b)/b$ is the square of a rational number n, whence $b=a^2/(4n^2)$. The proof is simplified by the remark that the quartic contains only even powers of x so that two of its roots are the negatives of two others. In view of the values of the y_i, we may set $x_2=-x_1$, $x_4=-x_3$. Then $x_1^2+x_3^2=-y_1=-2a$. But $x_1x_3=\sqrt{b}$. Hence $x_1^2-x_3^2=2\sqrt{(a^2-b)}$. Then

$$f\equiv\frac{x_1}{x_3}-\frac{x_3}{x_1}=2\sqrt{\frac{a^2-b}{b}}.$$

As above, f is rational only when $b=a^2/(1+n^2)$. Now f is numerically unaltered by (1324), but changed into its negative by (12), (34), (13)(24), or (14)(23).

To insure that G shall not reduce to a group of order 2, necessarily generated by (12)(34), (13)(24), or (14)(23), we add a condition to insure that the quartic shall be irreducible so that its group will be transitive. Since a^2-b is not the square of a rational number, the quartic $(x^2+a)^2=a^2-b$ has no linear factor. It will have the factors $x^2\pm cx+d$ if and only if $d=\pm\sqrt{b}$, $c^2=\pm2\sqrt{b}-2a\equiv(x_1\pm x_3)^2$. We thus assume that neither of the expressions $\pm2\sqrt{b}-2a$ is the square of a rational number.

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*Dickson's *Introduction to the Theory of Algebraic Equations* (Wiley & Sons), p. 56.

MISCELLANEOUS.

73. Proposed by CHARLES E. MYERS, Canton, Ohio (February, 1900. p. 64).

In an ice cream freezer, cream of a homogeneous character and at the uniform temperature of 60° F. is put into a cylinder having a closed base, and the whole put into a freezing mixture so as to subject the base and convex surface to a constant temperature of 30° F. Required the temperature at any point within the cream after the expiration of a given time.

Solution by F. H. SAFFORD, Ph. D., Instructor in Mathematics in the University of Pennsylvania.

In cylindrical coördinates the equation for the flow of heat is

$$D_t u = c^2 [D_r{}^2 u + \frac{1}{r} D_r u + \frac{1}{r^2} D_\phi{}^2 u + D_z{}^2 u]$$

[Byerly's *Fourier Series*, p. 1]. The axis of z is to be the axis of the cylinder. The planes of the upper and lower bases are respectively $z=0$ and $z=b$, and the radius of the cylinder is a. By the nature of the problem, the temperature u at any time t is independent of ϕ, so that the term above which involves ϕ is to be omitted. From Fourier's *Analytical Theory of Heat*, pp. 51, 113, $c^2 = K/CD$, where D is density of solid (cream), C is specific capacity, and K is specific conductibility.

Let k=temperature of medium surrounding sides and base, $k+m$=internal temperature when $t=0$; also let $v=m$ when $t=0$ ..... (1).

The differential equation becomes

$$D_t v = c^2 (D_r{}^2 v + \frac{1}{r} D_r v + D_z{}^2 v) \text{ ..... (2).}$$

In order to express the surface conditions due to the surrounding mixture we have two more differential equations

$$D_r u + h(u-k) = 0 \text{ when } r=a,$$
$$D_z u + h(u-k) = 0 \text{ when } z=b.$$

These become

$$D_r v + hv = 0 \text{ when } r=a \text{ ..... (3),}$$
$$D_z v + hz = 0 \text{ when } z=b \text{ ..... (4),}$$

and are explained in Fourier, p. 94. h is the constant of surface transmission. The value of v must satisfy conditions (1) ..... (4). Assume that $v=T.R.Z$ where T, R, and Z are functions of t, r, and z, respectively:

$$\therefore v = m \sum_{\kappa=1}^{\kappa=\infty} \sum_{\lambda=1}^{\lambda=\infty} \varepsilon^{-c^2 t(\mu_\kappa{}^2 + \rho_\lambda{}^2)} . A_\kappa J_0 (\mu_\kappa r) . B_\lambda \sin \rho_\lambda z \text{ .....(5).}$$

(For method, see Byerly, p. 195). J_0 is a Bessel Function while μ_κ and ρ_λ are

to be determined by (3) and (4) which give, after v is substituted from (5),

$$\mu_\kappa a J_1(\mu_\kappa a) - ahJ_0(\mu_\kappa a) = 0 \quad \ldots\ldots (6),$$
$$\rho_\lambda b \cos\rho_\lambda b + bh \sin\rho_\lambda b = 0 \quad \ldots\ldots (7).$$

It is shown in Riemann (*Partial Differential Equations*, §97) that (6) has an infinite number of real positive roots, while tables and a diagram in Gray and Mathew's *Bessel's Functions* make their calculation easy. Fourier, p. 270, states the same property for (7).

We use condition (1) in determing A_κ and B_λ, as when $t=0$ (5) must give

$$m = m \sum_{\kappa=1}^{\kappa=\infty} \sum_{\lambda=1}^{\lambda=\infty} A_\kappa J_0(\mu_\kappa r) B_\lambda \sin\rho_\lambda z.$$

It remains to obtain values of A_κ and B_λ such that

$$\sum_{\kappa=1}^{\kappa=\infty} A_\kappa J_0(\mu_\kappa r) = 1, \qquad \sum_{\lambda=1}^{\lambda=\infty} B_\lambda \sin\rho_\lambda z = 1.$$

From Byerly, pp. 229, 121, we find

$$A_\kappa = \frac{2h}{a(h^2+\mu_\kappa{}^2) J_0(\mu_\kappa a)} \quad \ldots\ldots (8),$$

$$B_\lambda = \frac{2b}{\rho_\lambda} \cdot \frac{\rho_\lambda{}^2 + h^2}{\rho_\lambda{}^2 b^2 + bh(bh+1)} (1 - \cos\rho_\lambda b) \quad \ldots\ldots (9).$$

The same excellent authority gives the means of solving cases in which the initial temperature is much more complicated than in the present problem.

In conclusion, the solution of the given problem is

$$u = k + m \sum_{\kappa=1}^{\kappa=\infty} \sum_{\lambda=1}^{\lambda=\infty} \varepsilon^{-c^2 t(\mu_\kappa{}^2 + \rho_\lambda{}^2)} . A_\kappa J_0(\mu_\kappa r) . B_\lambda \sin\rho_\lambda z,$$

the constants being determined by (6), (7), (8), (9).

PROBLEMS FOR SOLUTION.

ALGEBRA.

202. Proposed by G. B.M. ZERR, A. M., Ph. D., Parsons, W. Va.

Express in the form of radicals the roots of the equation

$$x^9+9mx^7+27m^2x^5+30m^3x^3+9mx^4x+2r=0.$$

203. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

The sum of a certain pair of roots of $x^4+ax^3+\left(\frac{2b}{a}+\frac{a^2}{4}\right)x^2+bx+c=0$ is equal to the sum of the remaining pair.

204. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

If α, β, γ be the roots of the cubic equation $x^3+qx+r=0$, prove that $3\Sigma\alpha^2\Sigma\alpha^5=5\Sigma\alpha^3\Sigma\alpha^4$.

GEOMETRY.

229. Proposed by F. D. POSEY, A.B., San Mateo, Cal., and G. W. GREENWOOD, B.A. (Oxon), Lebanon, Ill.

The solutions of problem 219 in the April number, "devise a simple geometric solution of the general quadratic equation," give the roots when they are *real*. Required a construction for the roots when they are *complex*.

230. Proposed by SAUL EPSTEEN, Ph. D., The University of Chicago.

Cut off a given area S from a gixen triangle ABC by means of a line passing through a given point P, (i) when P is on a side of ABC, (ii) when P is within ABC ($S<$area ABC). [For the case P outside of ABC see problem 218].

231. Proposed by B. F. FINKEL, A. M., Drury College, Springfield, Mo.

A man starts from the vertex, A, of a right isosceles triangle ABC, right-angled at A, and walks to D, the middle point of BC; from D to E, the middle point of AC; from E to F, the middle point of AD; from F to G, the middle point of DE; from G to H, the middle point of EF; from H to I, the middle point of FG; from I to J, the middle point of HI; and so on *ad infinitum*. Find the coördinates of his limiting position. [Suggested by Dr. Crawley].

232. Proposed by O. VEBLEN, Ph. D., The University of Chicago.

Given two parallel lines a_1, a_2, and two points A_1, A_2, upon a common perpendicular to a_1, a_2 such that A_1 is at the same distance distance from a_1 as A_2 is from a_2. Let P_1 be the foot of the perpendicular from a point P of the same plane to the line a_1 and P_2 the foot of the perpendicular from P to a_2. Find the locus of P when $\frac{PA_1}{PP_1}=\frac{PA_2}{PP_2}$.

CALCULUS.

180. Proposed by F. WENNER, Instructor in Physics, Iowa State College.

Solve $\frac{d^2y}{dx^2} + A\cos^2 y\frac{dy}{dx} + By = 0$.

MECHANICS.

169. Proposed by O. W. ANTHONY, Head of Mathematical Department, DeWitt Clinton High School, New York City.

A chain 12 feet long is supported by two pegs in the same horizontal plane 9 feet apart. If it be suddenly liberated from one of the pegs, find the time before the liberated end is directly below the other peg.

AVERAGE AND PROBABILITY.

155. Proposed by E. B. WILSON, Ph. D., Yale University.

The game of craps is played with two dice. If the player throws 7 or 11 on the first throw he wins. If he throws 12, 2, or 3 he loses. If the player throws any other number, that is to say, 4, 5, 6, 8, 9, 10, he is obliged to continue throwing until he throws that number again or until he throws 7. If he succeeds in throwing his first throw before he does 7, he wins—otherwise he loses. Required the odds against him. (Note that he can continue throwing indefinitely without getting either his original throw or the 7).

NOTES.

Professor Maxime Bôcher has been promoted to a full professorship at Harvard University. F.

Dr. Alexander Ziwet has been made a full professor of mathematics at the University of Michigan. F.

Dr. George H. Hallett has been made assistant professor of mathematics at the University of Pennsylvania. F.

Mr. W. H. Bursey and Mr. A. C. Lunn have taken the degree of doctor of philosophy at the University of Chicago.

BOOKS.

The Algebra of Invariants. By J. H. Grace, M. A., Fellow of Peterhouse and A. Young, M. A., Lecturer in Mathematics at Selwyn College, Late Scholar of Clara College. 8vo Cloth, vi+384 pages. Price, $3.00. Cambridge: The University Press. New York: The Macmillan Co.

The object of this work, as stated in the preface, is to provide an English introduction to the symbolical method in the theory of Invariants. The book contains sixteen chapters in which, after the first—the Introduction—the following subjects, in order, are treated: The Fundamental Theorem; Transvectants; Elementary Complete Systems; Gordan's Theorem; The Quintic; Simultaneous Systems; Hilbert's Theorem; Geometry; Apolarity and Rational Curves; Ternary Forms; Types of Covariants; and General Theorems on Covariants. The book concludes with an appendix in which is justified the results obtained by manipulating umbral expressions and in which is indicated how the whole theory can be made to rest on differential operators. Wronski's Theorem and Jordan's Lemma are also proved in the Appendix.

The first five chapters of the book lead gradually up to Gordan's proof of the finiteness of the system for a single binary form, while the sixth chapter is devoted to an exposition of Gordan's third proof.

While the authors have done valuable service to the cause of mathematics in making known to the English reader the methods of investigation in this branch of mathematics, of Gordan, Clebsch, and others of the German masters, yet it seems to me that the book could have been made more serviceable and better adapted to the use of a larger class of students, (i) by having presented the unsymbolic and symbolic methods together in the introduction, at least; (ii) by having used more illustrative examples in the development of the subject; and (iii) by having inserted a larger number of well chosen examples at the end of each chapter. F.

Elementary Geometry. Plane. By James McMahon, Assistant Professor of Mathematics in Cornell University. 8vo. Cloth, x+358 pages. New York and Chicago: The American Book Co.

This constitutes the sixth of the Modern Mathematical Series in preparation by the mathematical faculty of Cornell University and edited by Professor Wait. The book possesses many commendable features, notably the introduction of elementary ideas in Logic comparatively early, the arrangement of theorems and problems in natural groups and subgroups with reference to their underlying principles, and the purely geometrical treatment of ordinary size-relations. F.

School Algebra. By J. M. Colaw, A. M., Monterey, Va., and J. K. Ellwood, A. M., Principal of Colfax School, Pittsburg, Pa. 8vo. Cloth, 456 pages. Price, $1.15. Richmond, Va.: B. F. Johnson Publishing Co.

This book is well adapted to the needs of high schools and academies, and covers the field with sufficient thoroughness to meet the entrance requirements of any college or university. F.

Lessons in Physics. By Lothrop D. Higgins, Ph. B., Instructor in Natural Science in the Morgan School, Clinton, Conn. 12mo. Cloth, vii+379 pages. List Price, 90 cents. New York and Chicago: American Book Co.

This work presents a comprehensive and accurate view of Physics and is well suited to those schools having little or no laboratory facilities. F.

Elements of the Theory of Integers. By Joseph Bowden, Ph. D., Professor of Mathematics in Adelphi College, Brooklyn, N. Y. x+258 pages, 8vo. Cloth. Price, $1.25. New York: The Macmillan Co.

This volume, we are told in the preface, contains the first two parts of a work which the author hopes to complete, on the Elements of the Theory of Numbers, the remaining three parts to be devoted to Rational Numbers, Real Numbers, and Complex Numbers, respectively. The merits of this book are greatly obscured by the introduction and use of a cumbersome notation. The author also uses the "reform spelling." F.

Advanced Course in Algebra. By Webster Wells, S. B., Professor of Mathematics in the Massachusetts Institute of Technology. 8vo. Half Leather Back, viii+581 pages. Boston: D. C. Heath & Co.

This book develops the subject along the line followed in the author's College Algebra, though numerous improvements have been introduced. In the development of the fundamental laws of Algebra for positive and negative integers, the positive and negative fraction and zero, the most approved method of presentation has been used, and in general treatment the work will satisfy the thorough and critical teacher. F.

Laboratory Physics. A Student's Manual of a Series of Quantitative Experiments. By Dayton Clarence Miller, Professor of Physics in the Case School of Applied Science. Quarto, Cloth, vx+404 pages. Price, $2.00. New York, Boston, and Chicago: Ginn & Co.

This manual contains a very full course in laboratory work in Physics designed for undergraduates in colleges and technical schools. The explanations are full and clear and outline in a systematic manner those details of manipulation which benefit the student and are great aids to teachers. The number and variety of the problems are unusually large and well selected. F.

An Elementary American History. By D. C. Montgomery. 12mo Cloth, xiii+306 pages. List Price, 75 cents. Ginn & Co., Publishers.

A text book to meet the demand for a brief, continuous, narrative history of our country suited to the wants of elementary pupils. As in the author's Beginner's American History, every prominent topic is carefully and fully illustrated with pictures or maps, and the book appeals to the eye as well as the understanding. W. R.

Primary Arithmetic. By David Eugene Smith. 12mo Cloth, 264 pages. List Price, 30 cents. Ginn & Co., Publishers.

An excellent book in its arrangement and selection of problems. The illustrations and pictures will aid materially in making the work attractive and interesting. W. R.

Plane Trigonometry. By James M. Taylor, A. M., LL. D., Professor of Mathematics, Colgate University. 12mo. Cloth, viii+171 pages. List Price, 75 cents. Boston: Ginn & Co.

The treatment of the subject of Plane Trigonometry as here presented is clear, practical, and scientific. In addition to the usual matter contained in such a work, we have in this one a brief discussion of Hyperbolic Functions, Complex Numbers, and De Moivre's Theorem. F.

Technical Mechanics. By Edward R. Maurer, Professor of Mechanics in the University of Wisconsin. 8vo. Cloth, xvi+382 pages. New York: John Wiley & Sons.

This work is called a Technical Mechanics in accordance with the usage of some German writers. The theory in each chapter is grouped together and separated from the application. In Statics coextensive use is made of the graphical and algebraic methods. The book is well written, the treatment being lucid and the arrangement of material logical. It will be found to be of great service to the student of Engineering. F.

The Jones Series of Readers. Embracing the First, Second, Third, Fourth, and Fifth Readers. By L. H. Jones, A. M., President of the Michigan State Normal College, Superintendent of Schools of Indianapolis, Ind., and Cleveland, O. Boston: Ginn & Co.

In the first part of the First Reader are a number of attractive pictures in colors, a new feature in such books. The other books of the series are beautifully illustrated, and the printing and binding are excellent. The Fifth Reader contains some of the finest selections in the English Language. F.

A Student's Manual of a Laboratory Course in Physical Measurements. By Wallace Clement Sabine, A. M., Instructor in Harvard University. 8vo. Cloth, ix+126 pages. Boston: Ginn & Co.

This book contains the experiments performed in the course in Physics given in the Summer School of Harvard University. The course presupposes a knowledge of algebra, plane geometry, and the notation of trigonometry, and should be preceded by a more elementary course in physics. F.

One Thonsand Problems in Physics. By William H. Snyder, A. M., Master in Science, Worcester Academy, Worcester, Mass., and Irving O. Palmer, A. M., Master in Newton High School, Newton, Mass. 8vo. Cloth, v+142 pages. Boston: Ginn & Co.

This book contains a fine collection of elementary problems in physics. F.

Elementary Physical Geography. By William Morris Davis, Sturgis-Hooper Professor of Geology in Harvard University. 8vo. Cloth, xvii+401 pages+9 plates. Price, $1.25. Boston: Ginn & Co.

In this book the High School requirements are admirably satisfied. It certainly ranks as one of the best books that has yet appeared on the subject. F.

ERRATA.

Page 56, l. 6, for $(2c)!$ read $[(2c)!]^2$; l. 7, for 1 read $\sqrt{-1}$.

Page 58, l. 5, read $\int_0^\infty e^{-[x^2+(a^2/x^2)]\cos\theta}\cos\{[x^2+(a^2/x^2)]\sin\theta\}dx$.

Page 59, l. 4, for $-q$ read $+q$.

Page 59, l. 3 from bottom, for $=0$ read $=1$.

Page 60, l. 6 from bottom, for $\sqrt{-2}$ read $\sqrt{-1}$.

Page 61, l. 3, read $\int_0^\infty \left(\frac{e^{-ax}-e^{-bx}}{x}\right)^2\cos^2 rx dx$.

Page 62, l. 6 from bottom, for a_2 read a_1 in denominator.

Page 70, l. 16 from bottom, for T_1, T_2, T_2 read T_1, T_2, T_3; l. 8 from bottom, for $P=T_2$ read $P=T_1$.

Page 71, l. 12, for 2^2 read $2^2 m$; l. 14, for $(P+P)$ read $(P+Q)$; l. 18, for 2^{2n} read 2^n.

8–9

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SPHERICAL GEOMETRY.

By EDWIN BIDWELL WILSON.

LECTURE VII. CIRCLES, ROTATIONS, AND TANGENTS.

The properties of circles on the sphere are in a very large measure analogous to the properties of circles in the plane. There are, however, a number of minor differences which we shall touch upon; there are some things that are new; and there are a considerable number of theorems and points of view which are not taken up in the ordinary treatment. It is especially on account of these last that we take up the subject.

The ordinary definition of a circle as the locus of points at a given distance from a fixed point called the center would apply equally well to the sphere if we had developed the theory of distance.* But as we are not yet in possession of any measure of distance the definition may better be stated in another form as: The circle is the locus of all possible positions of one end of a given segment when the other end remains fixed at a point called the center of the circle. Or, if the idea of rotation is brought in, *the circle is the locus described by one end of a segment which is rotated about the other end as center.* The segment in any of its positions is called the radius of the circle. It may be noted that the point antipodal to the center also possesses all the properties of the center, and

*Throughout the following pages we speak of the sum and difference of segments. These words are used in a purely geometrical sense and their use may be justified in a manner such as that given on page 127. To think of the sum of two segments as a segment whose length is the sum of the lengths of the two segments is not necessary. It is not even advisable and until we have developed the theory of measure it is impossible. For this reason we use the term *congruent* in place of *equal*.